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BLOW-UP IN A SUBDIFFUSIVE MEDIUM OF INFINITE EXTENT

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Abstract

Thermal blow-up in a subdiffusive medium with a localized energy source is examined for a spatial domain of infinite extent in one, two, and three dimensions. An analysis of a nonlinear model of this problem reveals that a blow-up always occurs, independent of the spatial dimension and the thermal properties of the material. This behavior is in contrast with both classical diffusion and superdiffusion, where the prevention of a blow-up depends upon spatial dimension as well as the thermal properties of the medium. The asymptotic growth of the temperature near blow-up is determined for energy sources whose output increases in either an algebraic or exponential manner.

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1. Introduction

The problem of a thermal blow-up in a subdiffusive medium is examined within the framework of a fractional diffusion equation. The possibility of a blow-up is created by placing a localized, high-energy source at the origin of an unbounded medium with subdiffusive properties. A mathematical model of this blow-up problem is analyzed for a spatial domain of infinite extent in one, two, and three dimensions. It will be shown that a blow-up always occurs, independent of both the spatial dimension and the thermal properties of the material. This behavior is in contrast with both classical diffusion and superdiffusion, where the prevention of a blow-up depends upon spatial dimension as well as the thermal properties of the medium.

The results found here are consistent with the physical interpretation of a subdiffusive medium as described in [8, 9, 10, 11]. In materials with subdiffusive properties, the ability to diffuse heat is diminished. This allows for a build-up of thermal energy near a nonlinear source that leads to thermal runaway.

A related result for subdiffusive materials was found in [13], which examined blow-up in a one-dimensional strip of finite length. It was found that a blow-up could not be avoided even by placing the localized source arbitrarily close to a cold boundary. That behavior is consistent with the one-dimensional case considered here, where the cold boundaries have been placed infinitely far from the localized energy source. Other work on superdiffusive blow-up using different models and methods can be found in [6, 7].

Recent results in [14] for a superdiffusive medium of infinite extent revealed that a blow-up is influenced by spatial dimension. While a blow-up always occurs in one-dimension, it was found that in two and higher-dimensions that a blow-up could be avoided by enhancement of the superdiffusive properties of the medium. A comparison of the results here with those of [14] emphasizes the effectiveness of Lévy jumps in promoting thermal diffusion.

The model of subdiffusion used here follows that described in [2, 3, 5, 10, 11, 20] where the heat operator associated with classical diffusion is replaced by one with a fractional derivative. The ensuing partial differential equation for the temperature also includes a source term that models a localized energy source whose output grows nonlinearly with increasing temperature.

2. Mathematical formulation

It is assumed that the temperature $T(\mathbf{x},t)$, $\mathbf{x} = (x_1, \ldots, x_N)$, $N = 1, 2, 3$, in the unbounded spatial domain $I\!R^N$ of subdiffusive material satisfies the fractional diffusion equation

$$\frac{\partial T(\mathbf{x},t)}{\partial t} = \sum_{n=1}^{N} \frac{\partial^2}{\partial {x_n}^2} D_t^{1-\alpha}[T(\mathbf{x},t)] + \lambda D(\mathbf{x}|\mathbf{0})\, g[T(\mathbf{0},t)],\ \mathbf{x} \in I\!R^N, t > 0, \lambda > 0, \tag{1}$$

subject to the constraints

$$\begin{aligned} T(\mathbf{x},t) &\rightarrow 0, \text{ as } |\mathbf{x}| \rightarrow \infty, \quad t>0, \\ T(\mathbf{x},0) &= 0, \quad \mathbf{x} \in I\!R^N. \end{aligned} \tag{2}$$

The fractional derivative operator $D_t^{1-\alpha}$ is defined by

$$D_t^{1-\alpha}[T(\mathbf{x},t)] \equiv \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-t')^{\alpha-1} T(\mathbf{x},t')dt', \quad 0<\alpha<1, \tag{3}$$

where $\Gamma(\alpha)$ is the Gamma function. The anomalous diffusion parameter α governs particle mobility in subdiffusive material. The limiting case of $\alpha = 1$ is associated with classical diffusion.

The particular form of the operator (3) is known as the Riemann-Liouville fractional derivative, as discussed in [2, 5, 4, 10, 11, 15, 20]. For $\alpha = 1$, (3) reduces to the identity operator.

The source term in (1) is composed of a localization function $D(\mathbf{x}|\mathbf{0})$ and a nonlinear output function $g(T)$. The localization function has its support confined to a small region Ω centered about the origin such that

$$D(\mathbf{x}|\mathbf{0}) = \begin{cases} 1, & \mathbf{x} \in \Omega \\ 0, & \mathbf{x} \notin \Omega, \end{cases} \tag{4}$$

where

$$\Omega: \quad -a < x_n < a, \quad n = 1, \ldots, N, \quad N = 1,2,3. \tag{5}$$

Here it is assumed that $0 < a \ll 1$, so that the effective zone of the source is strongly localized near the origin. The nonlinear output function is assumed to have the properties

$$g(T) > 0, \quad g'(T) > 0, \quad g''(T) > 0, \quad T \geq 0. \tag{6}$$

This behavior for $g(T)$ provides for the possibility for a blow-up solution of (1)-(5). Such behavior occurs, for example, when $g(T) = \exp(T)$ is used as the model for an Arrhenius-type energy release in combustion theory.

The parameter $\lambda > 0$ arises from the process of nondimensionalization that yields (1)-(5). This parameter can be regarded as directly proportional to the strength of the source, and inversely proportional to the magnitude of the generalized diffusion constant.

3. Conversion to integral equation

It is advantageous to convert the nonlinear initial-boundary value problem given by (1)-(5) to an equivalent integral equation. This will be accomplished through the use of the free-space Green's function that satisfies

$$\frac{\partial G_\alpha(\mathbf{x},t|\boldsymbol{\xi},0)}{\partial t} = \sum_{n=1}^{N} \frac{\partial^2}{\partial {x_n}^2} D_t^{1-\alpha}[G_\alpha(\mathbf{x},t|\boldsymbol{\xi},0)] + \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t), \ \mathbf{x} \in I\!R^N, t > 0^-, \tag{7}$$

subject to the constraints

$$\begin{aligned} G_\alpha(\mathbf{x},t|\boldsymbol{\xi},0) &\rightarrow 0, \quad \text{as } |\mathbf{x}| \rightarrow \infty,\ t > 0, \\ G_\alpha(\mathbf{x},0^-|\boldsymbol{\xi},0) &= 0, \quad \mathbf{x} \in I\!R^N. \end{aligned} \tag{8}$$

Results presented in [21] imply that the solution of (7)-(8) can be expressed in terms of the solution for the classical diffusion case when $\alpha = 1$. That is

$$G_\alpha(\mathbf{x},t|\boldsymbol{\xi},0) = \int_0^\infty f_\alpha(\zeta) G_1(\mathbf{x}, t^\alpha \zeta|\boldsymbol{\xi},0) d\zeta. \tag{9}$$

To define $f_\alpha(\zeta)$, it is convenient to introduce the Mellin transform defined as

$$M[v(\zeta); z] \equiv \int_0^\infty \zeta^{z-1} v(\zeta) d\zeta. \tag{10}$$

In [21], $f_\alpha(\zeta)$ is the inverse Mellin transform given by

$$f_\alpha(\zeta) = M^{-1}\Big[\frac{\Gamma(z)}{\Gamma(1-\alpha+\alpha z)}\Big] = \sum_{j=0}^{\infty} \frac{(-1)^j \zeta^j}{j!\Gamma(1-\alpha-\alpha j)}, \quad \zeta \geq 0. \tag{11}$$

It follows that

$$\begin{aligned} f_\alpha(\zeta) \geq 0 \quad &\text{for} \quad \zeta \geq 0, \\ f_\alpha \rightarrow 0 \quad \text{exponentially as} \quad &\zeta \rightarrow \infty. \end{aligned} \tag{12}$$

The free-space Green's function for classical diffusion is given by

$$G_1(\mathbf{x},t|\boldsymbol{\xi},s) = \frac{H(t-s)}{[4\pi(t-s)]^{N/2}} \exp\Big[\frac{-1}{4(t-s)} \sum_{n=1}^{N} (x_n - \xi_n)^2\Big], \quad N = 1,2,3. \tag{13}$$

It follows from (1)-(2) and (7)-(8) that for $\mathbf{x} \in I\!R^N,\ t \geq 0$,

$$T(\mathbf{x},t) = \lambda \int_0^t \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G_\alpha(\mathbf{x}, t-s|\boldsymbol{\xi}, 0) D(\boldsymbol{\xi}, 0) g[T(\mathbf{0}, s)] d\xi_1 \cdots d\xi_N \, ds. \tag{14}$$

It is clear from (14) that if $T(\mathbf{0},t)$ is known, then $T(\mathbf{x},t)$ is determined for all $\mathbf{x} \in I\!R^N,\ t \geq 0$. Moreover, (14) implies that any blow-up solution of (1)-(5) must be associated with a blow-up of $T(\mathbf{0},t)$.

In order to determine $T(\mathbf{0},t)$, set $\mathbf{x} = \mathbf{0}$ in (14) to produce the nonlinear Volterra integral equation

$$u(t) = \int_0^t k_N(t-s) g[u(s)] ds, \quad t \geq 0, \tag{15}$$

where

$$u(t) \equiv T(\mathbf{0}, t). \tag{16}$$

The kernel k_N follows from (9), (13), and (14) as

$$\begin{aligned} k_N(t) &= \lambda \int_{-a}^{a} \cdots \int_{-a}^{a} G_\alpha(\mathbf{0}, t|\boldsymbol{\xi}, 0) d\xi_1 \cdots d\xi_N \\ &= 2^N \lambda \int_0^\infty f_\alpha(\zeta) \int_0^a \cdots \int_0^a G_1(\mathbf{0}, t^\alpha \zeta|\boldsymbol{\xi}, 0) d\xi_1 \cdots d\xi_N d\zeta \\ &= \frac{\lambda}{[\pi t^\alpha]^{N/2}} \int_0^\infty f_\alpha(\zeta) \zeta^{-N/2} \Big[\int_0^a \exp(\frac{-\xi^2}{4t^\alpha \zeta}) d\xi \Big]^N d\zeta. \end{aligned} \tag{17}$$

The integral of the exponential in (17) can be expressed as an error function that yields

$$k_N(t) = \lambda \int_0^\infty f_\alpha(\zeta) \Big[\mathrm{erf}(\frac{a}{2t^{\alpha/2}\zeta^{1/2}}) \Big]^N d\zeta. \tag{18}$$

The analysis of (15) for a possible blow-up solution requires some detailed knowledge about the properties of $k_N(t)$. In view of (12) and (18), it is clear that

$$k_N(t) \geq 0, \quad t \geq 0. \tag{19}$$

The non-increasing property of $k_N(t)$ follows by differentiation of (18), which gives

$$k_N^{'}(t) = -\frac{\alpha a \lambda N}{4t^{\frac{\alpha}{2}+1}} \int_0^\infty f_\alpha(\zeta) \zeta^{-1/2} \Big[\mathrm{erf}(\frac{a}{2t^{\alpha/2}\zeta^{1/2}}) \Big]^{N-1} d\zeta \leq 0,\ t > 0. \tag{20}$$

To derive the asymptotic behavior of $k_N(t)$ as $t \to 0$ and $t \to \infty$, it is useful to employ an approach developed in [1]. This approach uses the Parseval formula for Mellin transforms to convert (18) into an integral along a vertical path in the complex z-plane. This yields

$$k_N(t) = \frac{\lambda}{2\pi i} \int_{c-i\infty}^{c+i\infty} M[f_\alpha(\zeta); 1-z] M[F(\theta\zeta); z] dz, \tag{21}$$

where

$$F(\theta\zeta) \equiv \{\mathrm{erf}[(\theta\zeta)^{-\frac{1}{2}}]\}^N, \quad \theta \equiv \frac{4t^\alpha}{a^2}. \tag{22}$$

From (11), it follows that

$$M[f_\alpha(\tau); 1-z] = \frac{\Gamma(1-z)}{\Gamma(1-\alpha z)}, \tag{23}$$

which is seen to be analytic for $Re(z) < 1$. It follows from the properties of Mellin transforms that

$$M[F(\theta\zeta); z] = \theta^{-z} M[F(\zeta); z]. \tag{24}$$

To determine the analyticity of $M[F(\zeta); z]$, it is noted that

$$\zeta^{z-1}F(\zeta) = \zeta^{z-1}[\mathrm{erf}(\zeta^{-\frac{1}{2}})]^N \sim \begin{cases} \zeta^{z-1}, & \text{as} \quad \zeta \to 0 \\ (\frac{2}{\sqrt{\pi}})^N \zeta^{z-1-\frac{N}{2}}, & \text{as} \quad \zeta \to \infty, \end{cases} \tag{25}$$

This implies that $M[F(\zeta); z]$ exists and is analytic for $0 < Re(z) < \frac{N}{2}$, and

$$M[F(\zeta); z] = \begin{cases} z^{-1}, & \text{as} \quad z \to 0 \\ \left(\frac{2}{\sqrt{\pi}}\right) \left(z - \frac{N}{2}\right)^{-1}, & \text{as} \quad z \to \frac{N}{2}. \end{cases} \tag{26}$$

Thus, (18) can be expressed as

$$k_N(t) = \frac{\lambda}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{4t^\alpha}{a^2}\right)^{-z} \frac{\Gamma(1-z)}{\Gamma(1-\alpha z)} M[F(\zeta); z] dz, \tag{27}$$

where the vertical path of integration lies within the strip of analyticity of the integrand. That is,

$$Re(z) = 0 < c < \min\{Re(z) = 1, Re(z) = \frac{N}{2}\}. \tag{28}$$

Following the approach in [1], the asymptotic behavior of (27) as $t \to 0$ is determined by displacing the vertical path of integration to the left so as to capture the contribution from the pole at $z = 0$. This gives

$$k_N(t) \sim \lambda \left(\frac{4t^\alpha}{a^2}\right)^{-z} \frac{\Gamma(1-z)}{\Gamma(1-\alpha z)}\bigg|_{z=0} = \lambda \quad \text{as} \quad t \to 0. \tag{29}$$

This result is valid for $N = 1, 2, 3$.

Again following the approach in [1], the asymptotic behavior of (27) as $t \to \infty$ is determined by displacing the vertical path of integration to the right so as to capture the contribution from the first encountered pole implied by the upper bound in (28). This will depend upon N, so that each of the cases $N = 1, 2, 3$ must be considered separately.

For $N = 1$, the vertical path of integration in (27) is displaced to the right so as to capture the contribution from the dominant pole at $z = 1/2$. Thus,

$$k_1(t) \sim \frac{2\lambda}{\sqrt{\pi}}\left(\frac{4t^\alpha}{a^2}\right)^{-z} \frac{\Gamma(1-z)}{\Gamma(1-\alpha z)}\bigg|_{z=1/2} = \frac{\lambda a}{\Gamma(1-\alpha/2)} t^{-\frac{\alpha}{2}}, \quad \text{as} \quad t \to \infty. \tag{30}$$

For $N = 2$, the poles implied by the upper bound in (28) coalesce into a second-order pole at $z = 1$. Near $z = 1$,

$$\Gamma(1-z)M[F(\zeta); z]] \sim (-\frac{1}{z-1})(-\frac{4}{\pi(z-1)}) = \frac{4}{\pi(z-1)^2}. \tag{31}$$

Thus, the contribution from the second-order pole gives

$$\begin{aligned} k_2(t) &\sim \frac{-4\lambda}{\pi}\frac{d}{dz}\left[\left(\frac{4t^\alpha}{a^2}\right)^{-z} \frac{1}{\Gamma(1-\alpha z)}\right]\bigg|_{z=1} \\ &\sim \frac{4\lambda}{\pi\Gamma(1-\alpha)}\left(\frac{4t^\alpha}{a^2}\right)^{-1} \log\left(\frac{4t^\alpha}{a^2}\right) \\ &\sim \frac{\lambda a^2 \alpha}{\pi\Gamma(1-\alpha)} t^{-\alpha} \log t \quad \text{as} \quad t \to \infty. \end{aligned} \tag{32}$$

For $N = 3$, the vertical path is displaced to the right in (27) so as to capture the contribution from the dominate pole at $z = 1$ arising from $\Gamma(1-z)$. Thus,

$$\begin{aligned} k_3(t) &\sim \lambda\left(\frac{4t^\alpha}{a^2}\right)^{-z} \frac{M[F(\zeta); z]}{\Gamma(1-\alpha z)}\bigg|_{z=1} \\ &= \left[\frac{\lambda a^2}{4\Gamma(1-\alpha)}\int_0^\infty [\operatorname{erf}(\zeta^{-1/2})]^3 d\zeta\right] t^{-\alpha} \quad \text{as} \quad t \to \infty. \end{aligned} \tag{33}$$

4. Blow-up solution

The non-existence of a global solution to (15) and hence to (1)-(5) is associated with the blow-up behavior

$$u(t) \to \infty \text{ as } t \to \hat{t} < \infty. \tag{34}$$

Results for the existence and non-existence of global solutions to nonlinear Volterra integral equations has been developed and reviewed in [12, 16, 18, 19]. Those results are based on the relationship between a measure of the diffusive ability of the medium and a measure of the strength of the nonlinear energy source.

A measure of the subdiffusive medium's ability to transfer heat is given by $I_N(t)$, which is defined by

$$I_N(t) \equiv \int_0^t k_N(s)ds, \quad t \geq 0. \tag{35}$$

A measure of the strength of the energy output function $g(u)$ is given by either Λ or κ, which are defined by

$$\Lambda \equiv \sup_{0 \leq u < \infty} \left[\frac{u}{g(u)}\right] \leq \kappa \equiv \int_0^\infty \frac{du}{g(u)} < \infty. \tag{36}$$

The properties of $g(u)$ provided by (6) insure that $\Lambda \leq \kappa$, and it is assumed that $g(u)$ is such that $\kappa < \infty$.

The essential results on existence, uniqueness, and blow-up of the solution to (15) were summarized in [13] by the following two lemmas:

Lemma 1. *Let $k_N(t) \geq 0$ be continuous for $0 < t < \infty$ and integrable as $t \to 0$. Then (15) has a unique continuous solution for $0 \leq t < t^*$, where $t^* < \infty$ if t^* is such that*

$$I_N(t^*) = \Lambda, \tag{37}$$

while $t^* = \infty$ if

$$I_N(t) < \Lambda, \quad 0 \leq t < \infty. \tag{38}$$

Lemma 2. *Let $k_N(t) \geq 0$ be continuous and nonincreasing for $0 < t < \infty$ and integrable as $t \to 0$. Then whenever there exists a $t^{**} < \infty$ such that*

$$I_N(t^{**}) = \kappa, \tag{39}$$

it follows that (15) cannot have a continuous solution for $t \geq t^{**}$.

Bounds on the blow-up time $\hat{t}$ are implied by the lemmas as

$$0 < t^* \le \hat{t} \le t^{**} < \infty. \tag{40}$$

In order to apply *Lemma 1* and *Lemma 2*, it is essential to know the properties of $I_N(t)$. From the properties of $k_N(t)$ expressed by (19) and (29), it follows that $I_N(t)$ is continuous for $0 \le t < \infty$ and

$$I_N(t) > 0, \quad I'_N(t) > 0, \quad t > 0, \quad I_N(0) = 0. \tag{41}$$

From (29) follows the asymptotic behavior

$$I_N(t) \sim \lambda t \quad \text{as } t \to 0, \quad N = 1, 2, 3. \tag{42}$$

The asymptotic behavior of $I_N(t)$ as $t \to \infty$ plays a critical role in determining whether or not (15) has a blow-up solution. From the asymptotic behavior of $k_N(t)$ as $t \to \infty$ given by (30),(32), and (33), it follows that

$$I_N(t) \sim \begin{cases} \frac{\lambda a}{\Gamma(2-\frac{\alpha}{2})}\, t^{1-\frac{\alpha}{2}}, & N = 1, \\ \frac{\lambda a^2 \alpha}{\pi\Gamma(2-\alpha)}\, t^{1-\alpha} \log t, & N = 2, \\ \left[\frac{\lambda a^2}{4(\Gamma(2-\alpha)} \int_0^\infty [\mathrm{erf}(\zeta^{-1/2})]^3 d\zeta\right] t^{1-\alpha}, & N = 3, \qquad t \to \infty. \end{cases} \tag{43}$$

With the properties that have been derived for $I_N(t)$, it is now possible to state a theorem concerning the blow-up of the solution of (15).

Theorem 1. *For $N = 1, 2, 3$, the integral equation (15) has a unique continuous solution for $0 \le t < t^* < \infty$. That solution ultimately experiences a blow-up as $t \to \hat{t} < \infty$, where the bounds on $\hat{t}$ are provided by (40).*

P r o o f. The properties of $k_N(t)$ for $N = 1, 2, 3$ allow for the application of *Lemma 1* and *Lemma 2*. Since $I_N(0) = 0$ and $I_N(t) \to \infty$ as $t \to \infty$ for $N = 1, 2, 3$, then (37) must be satisfied by some t^* where $0 < t^* < \infty$. This establishes t^* as a lower bound on the extent of the existence and uniqueness of a continuous solution to (15). Moreover, since (39) must ultimately be satisfied by some $t^{**} < \infty$, then the solution of (15) must blow-up for some $\hat{t} \le t^{**} < \infty$. ∎

The physical implication of *Theorem 1* is that subdiffusive materials are unable to diffuse enough heat away from the nonlinear source to prevent a thermal runaway. This is true for $N = 1, 2, 3$, independent of any modification of the governing parameters of the problem.

5. Blow-up growth rate

The growth rate of the solution to (15) near blow-up can be determined through an asymptotic analysis as $t \to \hat{t}$. It is not necessary to know the exact value of $\hat{t}$ in order to do this analysis. In order to determine the asymptotic behavior of $u(t)$ near blow-up, it is appropriate to follow an approach developed in [17], which extended the concepts of [1] to obtain the asymptotic behavior near blow-up for a class of nonlinear Volterra equations that includes (15).

As shown in [17], the blow-up growth rate is determined by the behavior of $k_N(t)$ as $t \to 0$ and the behavior of $g(u)$ as $u \to \infty$. The asymptotic behavior of $k_N(t)$ as $t \to 0$ is given by (29). To determine the explicit behavior of $u(t)$ as $t \to \hat{t}$, it is necessary to provide explicit behavior of $g(u)$ as $u \to \infty$. The results here will be confined to the special cases in which $g(u)$ has either (i) algebraic growth or (ii) exponential growth as $u \to \infty$.

For the asymptotic analysis of (15), the method of [17] is followed by introducing the changes of variables

$$\eta = \left(\hat{t} - t\right)^{-1} - \eta_0, \quad \eta_0 = \hat{t}^{-1}, \quad w(\eta) = u(t). \tag{44}$$

This transformation converts (15) to the form ($0 \leq \eta < \infty$):

$$w(\eta) = \int_0^{\eta} k_N \left\{ (\eta - \eta') \left[(\eta' + \eta_0)(\eta + \eta_0) \right]^{-1} \right\} (\eta' + \eta_0)^{-2} \, g[w(\eta')] \, d\eta'. \tag{45}$$

In terms of the new variables, the blow-up defined by (34) is expressed as

$$w(\eta) \to \infty \text{ as } \eta \to \infty. \tag{46}$$

Following the method of [17], let $\eta' = \eta\tau$ so that (15) becomes

$$w(\eta) = \eta \, Q(\eta), \quad 0 \leq \eta < \infty, \tag{47}$$

where

$$Q(\eta) = \int_0^1 k_N \left\{ \eta(1-\tau)[(\eta\tau + \eta_0)(\eta + \eta_0)]^{-1} \right\} (\eta\tau + \eta_0)^{-2} \, g[w(\eta\tau)] \, d\tau. \tag{48}$$

Thus, the blow-up growth rate of $w(\eta)$ can be determined from an asymptotic analysis of (47) as $\eta \to \infty$. It is shown in [17] that the leading order behavior of $Q(\eta)$ as $\eta \to \infty$ requires only the leading order behavior of $k_N(t)$ as $t \to 0$. It then follows from (28) that

$$Q(\eta) \sim \lambda \int_0^\infty H(1-\tau)\,\Psi(\eta\tau)\,d\tau, \text{ as } \eta \to \infty, \tag{49}$$

where $H(1-\tau)$ is the Heaviside function and

$$\Psi(\eta\tau) \equiv (\eta\tau + \eta_0)^{-2}\, g[w(\eta\tau)]. \tag{50}$$

The integral in (49) can be converted to an integral in the complex z-plane by the application of the Parseval formula for Mellin transforms. This gives

$$Q(\eta) \sim \frac{\lambda}{2\pi i} \int_{c-i\infty}^{c+i\infty} M\,[H(1-\tau); 1-z]\, M[\Psi(\eta\tau); z]\, dz. \tag{51}$$

Further simplification of (51) is achieved by noting that

$$M\,[H(1-\tau); 1-z] = \frac{1}{1-z}, \tag{52}$$

and

$$M[(\Psi(\eta\tau); z] = \eta^{-z}\, M[\Psi(\tau); z]. \tag{53}$$

This allows the integral equation (47) to be replaced by the asymptotic equation

$$w(\eta) \sim \frac{\lambda}{2\pi i} \int_{c-i\infty}^{c+i\infty} \eta^{1-z}\, \frac{1}{1-z}\, M[\Psi(\tau); z]\, dz, \text{ as } \eta \to \infty. \tag{54}$$

To proceed with the asymptotic analysis, it is necessary to introduce the explicit behavior of $g(u)$ as $u \to \infty$:

(i) First consider the case in which $g(u)$ has algebraic growth,

$$g(u) \sim u^m, \quad m > 1, \text{ as } u \to \infty. \tag{55}$$

To obtain an asymptotic solution of (54) for this case, it is assumed that

$$u(\eta) \sim A\eta^p, \quad p > 0 \text{ as } \eta \to \infty. \tag{56}$$

The constants A and p are to be determined by satisfying (54) to leading order.

From (50), (55) and (56), it follows that

$$\Psi(\eta) \sim A^m \eta^{-2+mp} \text{ as } \eta \to \infty. \tag{57}$$

By imposing the restriction that $1 > 2 - mp$, it follows that $M[\Psi; z]$ has a simple pole at $z = 2 - mp < 1$, which dominates the pole at $z = 1$. Then

$$M[\Psi; r] \sim -\frac{A^m}{z - (2 - mp)} \text{ as } z \to 2 - mp, \tag{58}$$

since the leading asymptotic contribution from the integral in (54) comes from the pole implied by (58). As the vertical path of integration is displaced to the right to obtain the asymptotic behavior of (54) as $\eta \to \infty$, the pole implied by (58) provides the dominant contribution since it is encountered before the pole at $z = 1$. Thus (54) takes the form

$$A\,\eta^p \sim \frac{\lambda\, A^m}{mp - 1}\,\eta^{mp-1} \text{ as } \eta \to \infty. \tag{59}$$

From (59) it is concluded that

$$p = \frac{1}{m-1}, \quad A = \left[\frac{1}{\lambda\,(m-1)}\right]^{\frac{1}{m-1}}. \tag{60}$$

These results are seen to be consistent with the original constraint that $1 > 2 - mp = 1 - [1/(m-1)]$. The complement of this constraint, namely that the pole at $z = 1$ dominates that implied by (58), leads to a contradiction of any leading order asymptotic match in (54).

In view of (56) and (60), the asymptotic growth of the solution to (15) near blow-up is given by

$$u(t) \sim \left[\frac{1}{\lambda\,(m-1)\,(\hat{t} - t)}\right]^{\frac{1}{m-1}} \text{ as } t \to \hat{t}, \tag{61}$$

for the case in which $g(u)$ grows algebraically as specified by (55).

(ii) Next consider the case in which $g(u)$ has exponential growth,

$$g(u) \sim e^u \text{ as } u \to \infty. \tag{62}$$

To obtain an asymptotic solution of (54) for this case, it is assumed that

$$u(\eta) \sim \log\left(A\eta^p\right) \sim p\log\eta \text{ as } \eta \to \infty. \tag{63}$$

The constants A and p are to be determined by satisfying (54) to leading order.

From (50), (62) and (63), it follows that

$$\Psi(\eta) \sim A\eta^{-2+p} \text{ as } \eta \to \infty. \tag{64}$$

It follows that $M[\Psi; z]$ has a simple pole at $z = 2 - p$ and

$$M[\Psi; z] \sim -\frac{A}{z - (2 - p)} \text{ as } z \to 2 - p. \tag{65}$$

In order for the leading asymptotic contribution from the integral in (54) to yield a logarithmic term that will match (63), it is necessary that the simple pole for $M[\Psi; z]$ coalesces as $z \to 1$with the simple pole $-(z-1)^{-1}$. This coalescence requires that

$$p = 1. \tag{66}$$

As the vertical path of integration is displaced to the right, the leading order contribution from the double pole at $z = 1$ reduces (54) to

$$\log \eta \sim \lambda\, A\, \log \eta \text{ as } \eta \to \infty. \tag{67}$$

An asymptotic match in (67) is achieved by taking $A = \lambda^{-1}$, although this constant plays no role in the leading order behavior. In view of (63) and (66), the leading order asymptotic growth of the solution to (15) near blow-up is given by

$$u(t) \sim \log\left(\frac{1}{\hat{t} - t}\right) \text{ as } t \to \hat{t}, \tag{68}$$

for the case in which $g(u)$ grows exponentially as specified by (62).

5. Conclusion

An investigation of thermal blow-up in an unbounded subdiffusive medium results in *Theorem 1*, which assures that a blow-up always occurs, independent of the spatial dimension and the thermal properties of the material. This behavior is consistent with the physical interpretation of a subdiffusive medium as having a diminished capacity to diffuse heat.

The results here differ significantly from those of classical diffusion and superdiffusion. The classical diffusion version of (1), in which $\alpha = 1$, yields an $I_N(t)$ that converges as $t \to \infty$ when $N = 3$. A superdiffusion version of (1) considered in [14] yields an $I_N(t)$ that converges as $t \to \infty$ when $N \geq 2$. In each of these cases, the material properties of the medium can be modified to make λ sufficiently small such that (38) is satisfied and hence *Lemma 1* assures a non-blow-up solution of (1)-(5).

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